

Spectral Bundles and the DRY-Conjecture

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Abstract

Supersymmetric heterotic string models, built from a Calabi-Yau threefold X endowed with a stable vector bundle V , usually start from a phenomenologically motivated choice of a bundle V_v in the visible sector, the spectral cover construction on an elliptically fibered X being a prominent example. The ensuing anomaly mismatch between $c_2(V_v)$ and $c_2(X)$, or rather the corresponding differential forms, is often 'solved', on the cohomological level, by including a fivebrane. This leads to the question whether the difference can be alternatively realized by a further stable bundle. The 'DRY'-conjecture of Douglas, Reinbacher and Yau in math.AG/0604597 gives a sufficient condition on cohomology classes on X to be realized as the Chern classes of a stable sheaf. In 1010.1644 [hep-th] we showed that infinitely many classes on X exist for which the conjecture is true. In this note we give the sufficient condition for the mentioned fivebrane classes to be realized by a further stable bundle in the hidden sector. Using a result obtained in 1011.6246 [hep-th] we show that corresponding bundles exist, thereby confirming this version of the DRY-Conjecture.

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To get fourdimensional $N = 1$ supersymmetric models from the tendimensional $E_8 \times E_8$ heterotic string one compactifies on a Calabi-Yau threefold X endowed with a polystable holomorphic vector bundle $V' = (V_v, V_h)$. V_v is usually a stable bundle considered to be embedded in (the visible) E_8 (V_h plays the corresponding role for the hidden E_8).

Usually one specifies V_v to fulfill some phenomenological requirements, like the generation number. This specifies discrete parameters of the bundle construction and anomaly freedom of the construction is encoded in the integrability condition for the existence of a solution to the anomaly cancellation equation

$$c_2(X) = c_2(V_v) + W. \quad (0.1)$$

Here W , as it stands, has just the meaning to indicate a possible mismatch for a certain bundle V_v ; it can be understood either as the cohomology class of (the compact part of the world-volume of) a fivebrane, or as second Chern class of a further stable bundle V_h in the hidden sector. In the first case the class of W has to be effective for supersymmetry to be preserved. If one wants to solve without a fivebrane one has to make sure that a corresponding hidden bundle V_h with having $c_2(V_h) = W := c_2(X) - c_2(V_v)$ exists.

In [5] it has been shown that when (0.1) can be satisfied with $W = 0$ then X and V_v can be deformed to a solution of the anomaly equation even already on the level of differential forms.

This leads to the general question to give sufficient conditions for the existence of stable bundles with prescribed Chern class $c_2(V)$. Concerning this a conjecture is put forward in [1] by Douglas, Reinbacher and Yau (DRY) (actually we use the particular case of the conjecture with $c_1(V) = 0$). We will actually use a weaker version of the conjecture, considered already in [7]. We recall the following definition and conjecture.

Definition. Let X be a Calabi-Yau threefold of $\pi_1(X) = 0$ and $c \in H^4(X, \mathbf{Z})$,

- i) c is called a *Chern class* if a stable $SU(\mathcal{N})$ vector bundle V on X exists with $c = c_2(V)$
- ii) c is called a *DRY class* if an ample class $H \in H^2(X, \mathbf{R})$ exists (and an integer \mathcal{N}) with

$$c_2(V) = \mathcal{N} \left(H^2 + \frac{c_2(X)}{24} \right). \quad (0.2)$$

Weak DRY-Conjecture. On a Calabi-Yau threefold X of $\pi_1(X) = 0$ every DRY class $c \in H^4(X, \mathbf{Z})$ is a Chern class.

Here it is understood that the integer \mathcal{N} occurring in the two definitions is the same.

We choose X to be elliptically fibered over a rational base surface B with section $\sigma : B \rightarrow X$ (we will also denote by σ the embedded subvariety $\sigma(B) \subset X$ and its cohomology class in $H^2(X, \mathbf{Z})$), a case particularly well studied in phenomenological applications. Typical examples for B are Hirzebruch or del Pezzo surfaces (or suitable blow-ups of these). As in [7] we consider bases B for which $c_1 := c_1(B)$ is ample (this excludes the Hirzebruch surface \mathbf{F}_2). (The classes c_1^2 and $c_2 := c_2(B)$ will be considered as (integral) numbers.)

On X one has according to the general decomposition $H^4(X, \mathbf{Z}) \cong H^2(X, \mathbf{Z})\sigma \oplus H^4(B, \mathbf{Z})$

$$c_2(V) = \phi\sigma + \omega \quad (0.3)$$

where ω is understood as an integral number (pullbacks from B to X will be usually suppressed). Similarly one has $c_2(X) = 12c_1\sigma + c_2 + 11c_1^2$ or, with Noethers theorem, $12c_1\sigma + 10c_1^2 + 12$.

One gets the following theorem [7]

Theorem on DRY classes. *A class $c = \phi\sigma + \omega \in H^4(X, \mathbf{Z})$ is a DRY class if and only if the following condition is fulfilled (where b is some $b \in \mathbf{R}^{>0}$ and $\omega \in H^4(B, \mathbf{Z}) \cong \mathbf{Z}$):*

$\phi - \mathcal{N}(\frac{1}{2} + b)c_1$ is ample and $\frac{1}{\mathcal{N}}\omega > \omega_0(\phi; b) := r + \frac{c_1^2}{4}(b + \frac{q}{b})$.

Here one uses the following abbreviations, cf. [7]

$$r := \frac{1}{2\mathcal{N}}\phi c_1 + \frac{1}{6}c_1^2 + \frac{1}{2} \quad (0.4)$$

$$q := \frac{(\phi - \frac{\mathcal{N}}{2}c_1)^2}{\mathcal{N}^2 c_1^2}. \quad (0.5)$$

Note that under the hypothesis that $\phi - \frac{\mathcal{N}}{2}c_1 = A + b\mathcal{N}c_1$ with an ample class A on B (and with $c_1 \neq 0$ effective) one has $(\phi - \frac{\mathcal{N}}{2}c_1)^2 > b^2\mathcal{N}^2 c_1^2$, thus one has $b < \sqrt{q}$.

To make the condition more explicit let us choose a concrete b : the choice $b = 1/2$, for example, gives the conditions

$$\phi - \mathcal{N}c_1 \quad \text{ample} \quad (0.6)$$

$$\frac{1}{\mathcal{N}}\omega > \frac{5}{12}c_1^2 + \frac{1}{2} + \frac{1}{2\mathcal{N}^2}\phi^2. \quad (0.7)$$

These are the conditions for the application of the weak DRY conjecture. By contrast recall that the starting conditions of the effectivity of the fivebrane class $W = c$ were just

$$\phi \quad \text{effective} \quad (0.8)$$

$$\omega \geq 0. \quad (0.9)$$

In the case of our application, where c arises as $c_2(X) - c_2(V_v)$ for an $SU(n)$ spectral cover bundle V_v , one has for the effective fivebrane class $W = c$ the following

$$\phi = 12c_1 - \eta_v \quad (0.10)$$

$$\omega = 10c_1^2 + 12 + \frac{n^3 - n}{24}c_1^2 - (\lambda_v^2 - \frac{1}{4})\frac{n}{2}\eta_v(\eta_v - nc_1). \quad (0.11)$$

Here η_v is an effective class in B with $\eta_v - nc_1$ also effective and λ_v is a half-integer satisfying the following conditions: λ_v is strictly half-integral for n being odd; for n even an integral λ_v requires $\eta_v \equiv c_1 \pmod{2}$ while a strictly half-integral λ_v requires c_1 even. (Often one assumes that $\eta_v - nc_1$ is not only effective but even ample in B .)

As the visible bundle is a spectral cover bundle, and so is stable with respect to the typical spectral Kähler class (the polarization) $H = \epsilon H_0 + H_B$ (where H_0 is an ample class on X , ϵ is chosen sufficiently small [3] and H_B is an ample class on B) we have to choose a hidden bundle V_h which is stable, or polystable, with respect to the same class. This suggests to take for V_h also a spectral bundle (which we take with ample spectral cover surface). This will allow to produce in $c_2(V_h)$ the part $\phi\sigma$ of the needed class $W = c = \phi\sigma + \omega$. However the special form of the remaining fiber term of $c_2(V_h)$ for a spectral bundle V_h will usually not be able to represent a given ω part in c . Therefore one needs a second input to match also this part of c . Suitable to represent this part is a pull-back bundle π^*E (where E is a stable bundle on B) as it has $c_2(\pi^*E) = k$, and so consists just of the needed part where the integer k is arbitrarily choosable as long $k \geq r(E) + 2$ where $r(E) := \text{rank}(E)$ [8]. Therefore, in total, we will choose in the hidden sector a combination of a spectral $SU(N)$ bundle and a pull-back bundle π^*E of rank $r(E)$ such that $\mathcal{N} = N + r(E)$. It is possible to combine the advantages of a hidden spectral bundle with those of a hidden pullback bundle because, according to [6], the pullback bundle will be stable with respect to the same polarization as the spectral bundle.

Thus combining the flexibilities of the (hidden) spectral bundle in the ϕ -term (in the $c = \phi\sigma + \omega$ decomposition of $c_2(V_h) + c_2(\pi^*E)$; we call this the “ σ ”-term of c_2) with that of the the pullback bundle in the ω -term (which we call the “fiber”-term) we will be able, as shown below, to represent a class $c = \phi\sigma + \omega$ as c_2 of the total hidden (sum) bundle if it satisfies the following conditions (referred to below as conditions of “realizability” of the class c)

$$\phi - Nc_1 \quad \text{ample} \quad (0.12)$$

$$\omega \geq 1. \quad (0.13)$$

On the other hand, the conditions for a DRY class were

$$\phi - \mathcal{N}\left(\frac{1}{2} + b\right)c_1 \quad \text{ample} \quad (0.14)$$

$$\frac{1}{\mathcal{N}}\omega > r + \frac{c_1^2}{4}\left(b + \frac{q}{b}\right). \quad (0.15)$$

Thus one sees that for $N = 2 = r(E)$ all DRY classes are realised as second Chern classes of a polystable bundle $V_h \oplus \pi^*E$: for a DRY class one has that $\phi - \frac{\mathcal{N}}{2}c_1$ is ample, so the condition in (0.12) is satisfied (this works always as long as one has $\mathcal{N}/2 \geq N$, i.e. $r(E) \geq N$). Furthermore the necessary condition $\frac{1}{\mathcal{N}}\omega \geq r + s$ for the fiber part of a DRY class (cf. [7], here $s = \frac{1}{2\mathcal{N}}\sqrt{c_1^2}\sqrt{(\phi - \frac{\mathcal{N}}{2}c_1)^2}$) implies that

$$\omega \geq \frac{1}{2}\phi c_1 + \frac{\mathcal{N}}{6}c_1^2 + \frac{\mathcal{N}}{2} + \frac{1}{2}\sqrt{c_1^2}\sqrt{(\phi - \frac{\mathcal{N}}{2}c_1)^2}. \quad (0.16)$$

Thus this gives for $\mathcal{N} = 4$ that one has (0.13) for the fiber part of all DRY classes.

Thus we have shown the weak DRY conjecture for the rank 4 case, in the sense of realising a DRY class by the c_2 of a corresponding rank 4 bundle; however we have shown only a realisation by a polystable bundle which is a sum $V_h \oplus \pi^*E$. (Starting from this bundle one can build, as a variant of this construction, an irreducible, stable bundle, constructed as a non-split extension.)

It remains to check that all classes, as described in (0.12) and (0.13), are indeed realizable as $c_2(V_h) + c_2(\pi^*E)$. First notice that one has

$$c_2(V_h \oplus \pi^*E) = \eta_h \sigma - \frac{N^3 - N}{24} c_1^2 + \frac{N}{2} (\lambda_h^2 - \frac{1}{4}) \eta_h (\eta_h - N c_1) + k. \quad (0.17)$$

The σ term is clear, one takes $\eta_h := \phi$. The instanton number k will be arbitrarily specifiable [8] if one has $k \geq rk(E) + 2$. For the special case $r(E) = 2$ one has, however, the stronger result that $k \geq 2$ is sufficient, cf. [9], Ch. 10, Th. 37. So, to have arbitrary $\omega \geq 1$ realizable, we need to produce a V_h with the fiber term of $c_2(V_h)$ being ≤ -1 . Having $N = 2$ and taking the optimal possibility $\lambda_h = 0$ we get for this fiber part $-\frac{1}{4}c_1^2 - \frac{1}{4}\eta_h(\eta_h - 2c_1) = -\frac{1}{4}(\eta_h - c_1)^2$; furthermore the conditions in the spectral construction demand in this case that $\eta_h \equiv c_1 \pmod{2}$, i.e., $\eta_h = c_1 + 2a$ (for an integral class a , which is nonzero as $\eta_h - 2c_1$ has to be effective), thus leaving the term $-a^2$. Now one has that actually $a^2 > 0$: for one has $c_1^2 > 0$ and $\eta_h(\eta_h - 2c_1) > 0$ because $\eta_h - 2c_1$ and c_1 are here ample.

Often one will have the sharper $a^2 \geq 2$ which will allow $\omega \geq 0$ in (0.13): if c_1 is even, for example (because of $a^2 \equiv ac_1 \pmod{2}$), or on a Hirzebruch surface (where already $\frac{1}{4}c_1^2 = 2$).

Finally let us come back to the original problem which motivated our consideration of the DRY conjecture. Notice that, after our consideration of the question which classes c are realizable as Chern classes, the focus of the original question has actually somewhat shifted: it will be enough to investigate, whether the fivebrane class W belongs to these “realizable” classes; for note that in any case the DRY class condition is only sufficient for a class to be realizable as Chern class.

This demand of “realizability” means, concerning the σ part, whether $\phi = 12c_1 - \eta_v$, which is effective by assumption, fulfills actually the stronger condition that $\phi - 2c_1$ is ample; and concerning the fiber part, whether ω in (0.11), which by assumption is ≥ 0 , is actually > 0 .

Let us conclude: usually one supplemented a phenomenological heterotic spectral bundle construction, which was constructed in the visible sector, with a fivebrane class W (to solve the anomaly cancellation equation); then the only condition one had to satisfy was that this class $W = c = \phi\sigma + \omega$ is effective (i.e. ϕ effective and $\omega \geq 0$). Here we have seen that, sharpening this demand just slightly (to $\phi - 2c_1$ ample and $\omega > 0$) one can actually turn on a polystable bundle $V_h \oplus \pi^*E$ in the hidden sector whose c_2 just realizes the class W (furthermore the total bundle including also V_v is polystable). One can thus solve the anomaly for (almost) all the visible spectral bundles (for which one can solve it *with* fivebrane) also *without* any fivebrane (and instead just with a hidden bundle). Then one can actually solve the anomaly already on the level of differential forms [6].

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